Solution of Newton's equation for Earth's Gravitational Potential

$$F(x) = -\frac{GMm}{x^2}$$

$$x(t) = ?$$

Energy integral: $\displaystyle\int_{x_0}^{x} \frac{dx}{\pm\sqrt{E + \frac{GMm}{x}}} = \sqrt{\frac{2}{m}}\,t$ will lead to *x(t)*

For $E < 0$ energy integral becomes

$$\int_{x_0}^{x} \frac{dx}{\pm\sqrt{E + \frac{GMm}{x}}} = \sqrt{\tfrac{2}{m}}\,t$$

Setting $\cos\theta = \sqrt{\dfrac{-Ex}{GMm}}$ energy integral becomes:

$$\pm\frac{GMm}{(-E)^{3/2}} \int_{\theta_0}^{\theta} 2\cos^2\theta\, d\theta = \sqrt{\tfrac{2}{m}}\,t$$

This leads to a pair of parametric equations

$$t(\theta) = \pm\sqrt{\frac{x_{\max}^3}{2GM}}\left(\theta + \tfrac{1}{2}\sin 2\theta\right)\Big|_{\theta_0}^{\theta}$$

$$x(\theta) = x_{\max}\cos^2\theta$$

$$x_{\max} = \frac{GMm}{-E}$$

+ sign in Eq. for $t(\theta)$: when x decreases with increasing θ (descending object)
– sign in Eq. for $t(\theta)$: when x increases with increasing θ (ascending object)

By varying the parameter θ, a table may be generated for $x(\theta)$ and $t(\theta)$. Plots may then be created for $x(t)$. Examples for various situations are given in the next few pages.

Example 1: Object dropped from rest from x_{max} ($E < 0$)

$$v_0 = 0 \Rightarrow \theta_0 = 0$$

$$x_0 = x_{max}$$

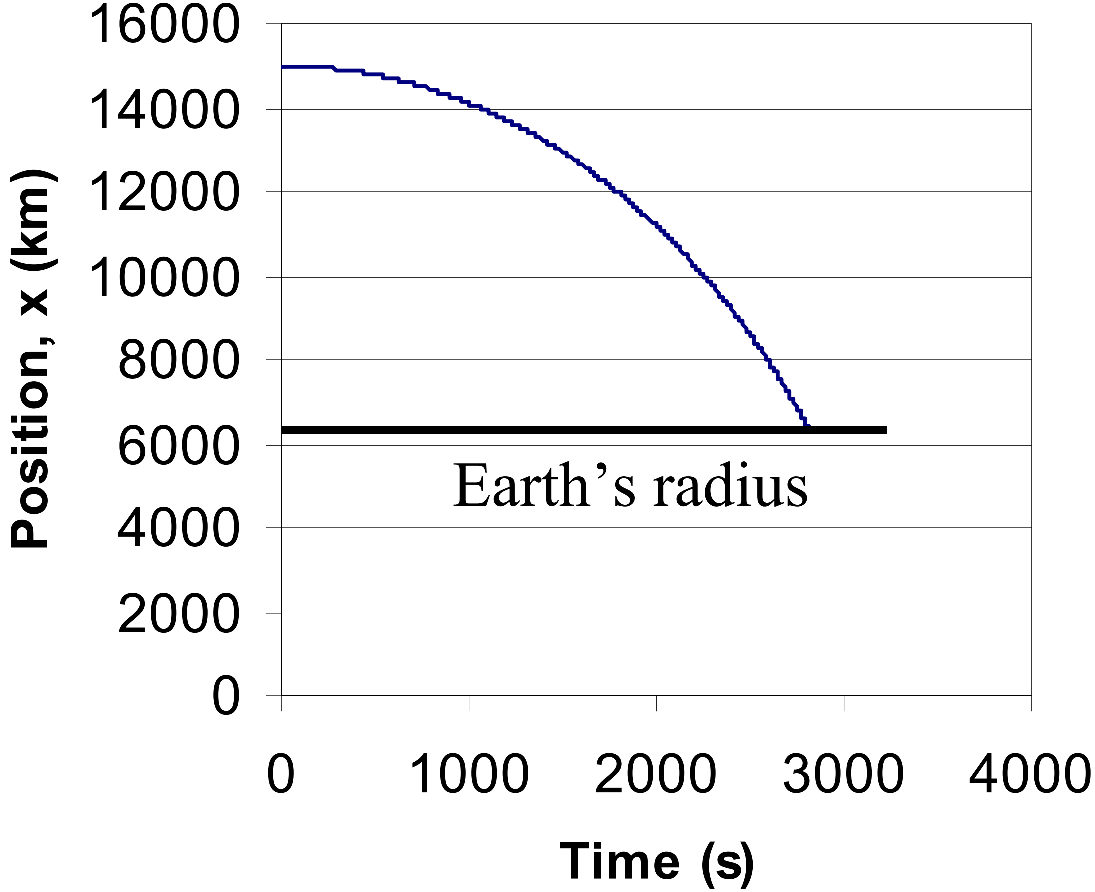
Leads to

$$t(\theta) = +\sqrt{\frac{x_{max}^3}{2GM}}\left(\theta + \tfrac{1}{2}\sin 2\theta\right)$$

$$x(\theta) = x_{max}\cos^2\theta$$

See plot next page

Drop from rest with x0 = xmax = 15,000 km



Example 2: Launch from Earth's surface with v_0 less than escape velocity ($E < 0$)

$$\left.\begin{aligned} v_0 &= 5.55 \ \text{km/s} \\ x_0 &= 6370\,\text{km} \end{aligned}\right\} \Rightarrow \theta_0 = \frac{\pi}{6}$$

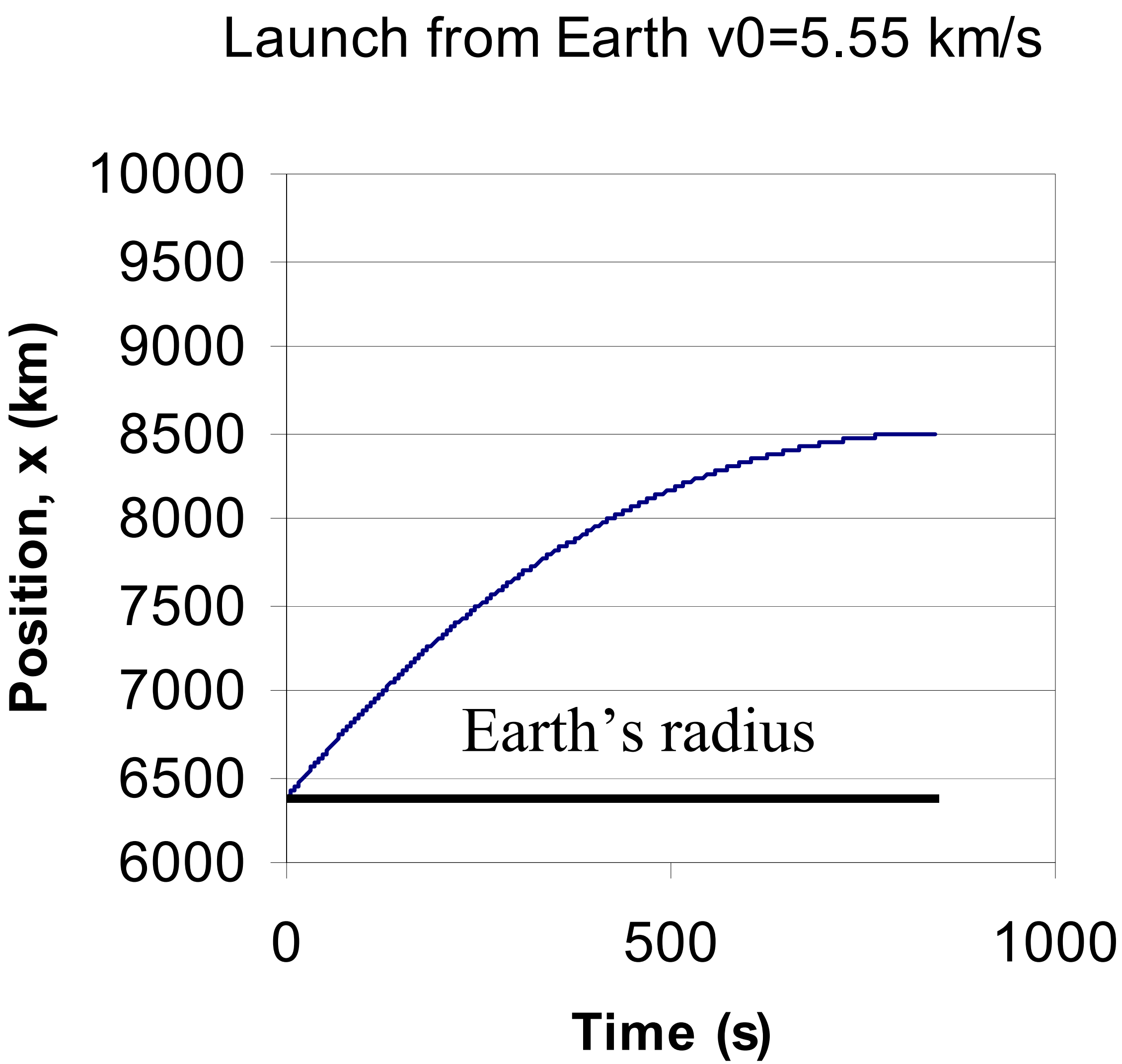
$$x_{\max} = \frac{GMm}{-E} = 8,493\,\text{km}$$

Leads to

$$t(\theta) = -\sqrt{\frac{x_{\max}^3}{2GM}}\left(\theta + \tfrac{1}{2}\sin 2\theta - \tfrac{\pi}{6} - \tfrac{1}{2}\sin\tfrac{\pi}{3}\right)$$

$$x(\theta) = x_{\max}\cos^2\theta$$

See plot next page



Launch from Earth v0=5.55 km/s
10000
9500
9000
8500
8000
7500
7000
6500
6000
Position, x (km)
Earth's radius
0
500
1000
Time (s)

Example 3: Launch from Earth's surface with large v_0 but still less than escape velocity ($E < 0$)

$$\left.\begin{aligned} v_0 &= 9.62 \text{ km/s} \\ x_0 &= 6370\,\text{km} \end{aligned}\right\} \Rightarrow \theta_0 = \frac{\pi}{3}$$

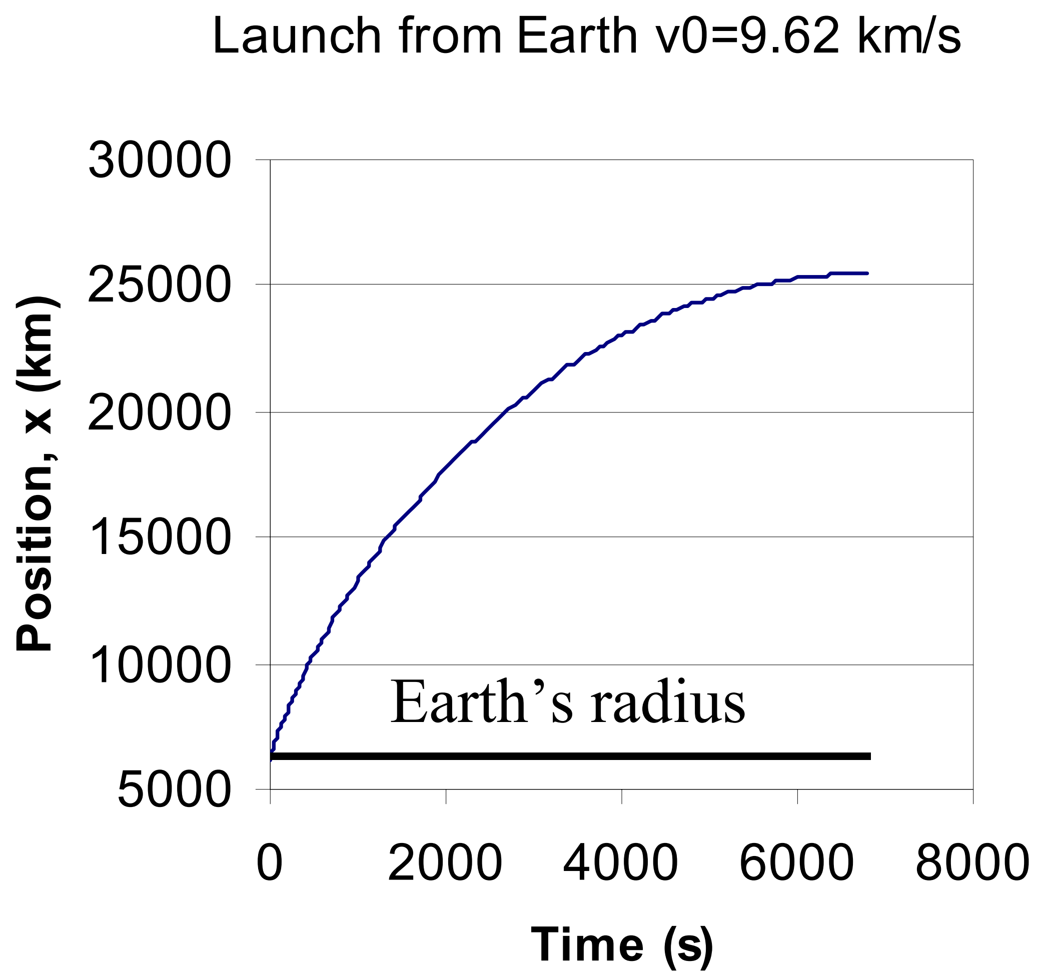
$$x_{\max} = \frac{GMm}{-E} = 25,480\,\text{km}$$

Leads to

$$t(\theta) = -\sqrt{\frac{x_{\max}^3}{2GM}}\left(\theta + \tfrac{1}{2}\sin 2\theta - \tfrac{\pi}{3} - \tfrac{1}{2}\sin\tfrac{2\pi}{3}\right)$$

$$x(\theta) = x_{\max}\cos^2\theta$$

See plot next page


Launch from Earth v0=9.62 km/s
30000
25000
20000
15000
10000
5000
Position, x (km)
Earth's radius
0
2000
4000
6000
8000
Time (s)

For $E > 0$ energy integral becomes

$$\int_{x_0}^{x} \frac{dx}{\pm\sqrt{E + \frac{GMm}{x}}} = \sqrt{\frac{2}{m}}t$$